On the asymptotic behaviour of the number of renewals via translated Poisson

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Some facts about renewal processes

- Let τ and τ_i's be independent and identically distributed inter-renewal times with mean μ and finite variance σ^2, define $S_n := \sum_{i=1}^{n} \tau_i$ and $S_0 = \sum_{i=1}^{0} \tau_i := 0$
- Let $N(t)$ be the number of renewals up to time t, that is,

$$N(t) = \max\{n : \; S_n \leq t < S_{n+1}\},$$

- $\mathbb{E}N(t) \sim \frac{t}{\mu}$ and $\mathrm{Var}(N(t)) \sim \frac{t\sigma^2}{\mu^3}$ as $t \to \infty$
  - Feller (1968), Vol 1, pp. 340–341; Vol. 2, p. 372.
- $W(t) := \frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}}$ is approximated by a standard normal distribution.
- **Question:** how fast?

Who cares?

- Renewal theory is one of indispensable topics in introductory courses of random processes.
- The behaviour of regenerative events is of significant interest in probability theory and related areas.

Speed of convergence?

- For the Kolmogorov distance, it must be easy!

$$\mathbb{P}(N(t) \geq n) = \mathbb{P}(S_n \leq t).$$

  - $S_n \sim N(n\mu, n\sigma^2)$.
  - Most likely values of $N(t)$ are within a few standard deviations of the mean, but the approximate normal is $N\left(\frac{t}{\mu}, \frac{t\sigma^2}{\mu^3}\right)$.
  - Do you still think it is easy?

Literature?

- Englund (1980):

$$\sup_n \left| \mathbb{P}(N(t) < n) - \Phi\left(\frac{(n\mu - t)\sqrt{\mu}}{\sigma\sqrt{t}}\right) \right| \leq 4\left(\frac{\gamma}{\sigma}\right)^3 \sqrt{\frac{\mu}{t}},$$

  where $\gamma^3 = \mathbb{E}(|\tau - \mu|^3)$.

  – $\Phi\left(\frac{(n\mu - t)\sqrt{\mu}}{\sigma\sqrt{t}}\right) = \mathbb{P}(X_t < n)$, with $X_t \sim N(t/\mu, t\sigma^2/\mu^3)$.

  – $d_{\mathrm{K}}(Q_1, Q_2) := \sup_{u \in \mathbb{R}} |Q_1(-\infty, u] - Q_2(-\infty, u]|$.

  – Under the Kol. distance, it is done!

  – Can someone use Stein's method to prove this bound?

More on literature

- Omey and Vesilo (2011): suppose the characteristic function of τ is integrable, then

$$\sup_t \left| \frac{\sigma\sqrt{n}}{\mu} \mathbb{P}(N(t) = n+1) - \Phi\left(\frac{t - n\mu}{\sigma\sqrt{n}}\right) \right| = o(1).$$

Moreover, if τ has finite third moment, then

$$\sup_t \left| \frac{\sigma\sqrt{n}}{\mu} \mathbb{P}(N(t) = n+1) - \Phi\left(\frac{t - n\mu}{\sigma\sqrt{n}}\right) \right| = O\left(n^{-1/2}\right).$$

- $\mathbb{P}(N(t) \in A)$ for any set $A \subset \mathbb{Z}_+$? Discretised normal?

Metrics?

How about the probabilities of values in any set?

- Total variation distance: for any probability measures Q_1 and Q_2 on $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$,

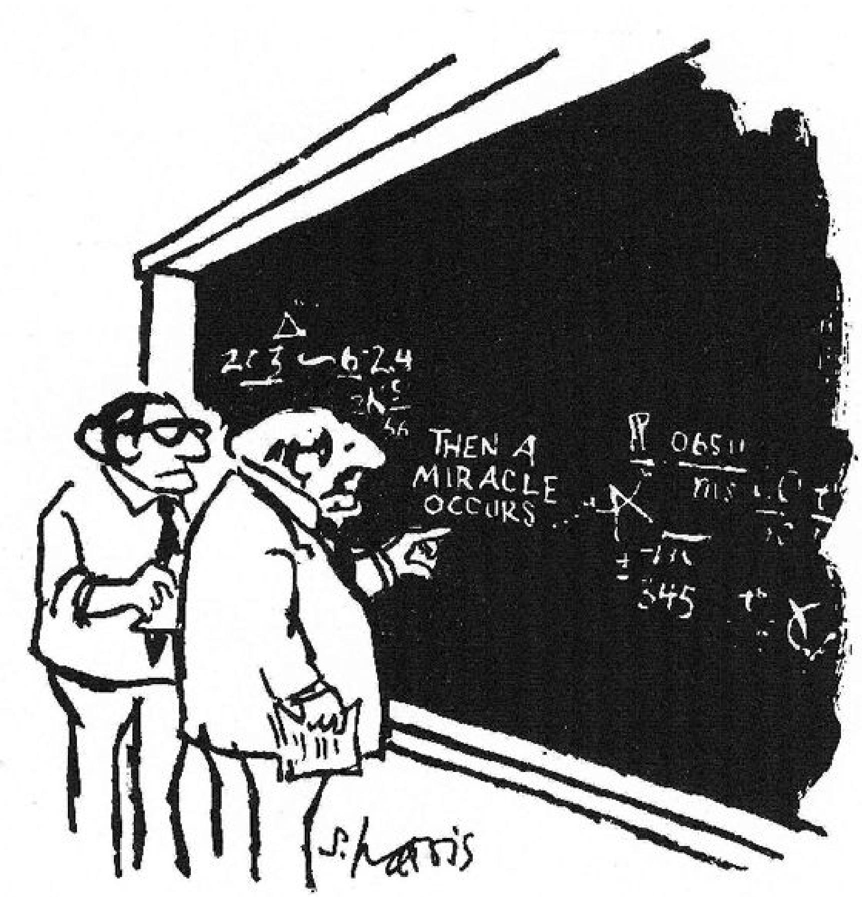
$$d_{\mathrm{TV}}(Q_1, Q_2) := \sup_{A \subset \mathbb{Z}} |Q_1(A) - Q_2(A)|.$$

- The Wasserstein distance: Q_1 and Q_2 on $\mathbb{R}$

$$d_{\mathrm{W}}(Q_1, Q_2) \quad := \quad \sup_{f \in \mathcal{F}_{\mathrm{Lip}(1)}} \left| \int f dQ_1 - \int f dQ_2 \right|,$$

where $\mathcal{F}_{\mathrm{Lip}(1)} = \{f : \mathbb{R} \mapsto \mathbb{R} : \ |f(x) - f(y)| \leq |x - y| \text{ for all } x, y \in \mathbb{R}\}$.

By handwaving: it should work!



"I think you should be more explicit here in step two."

- The cartoon is by Sidney Harris

How to do it?

- Characteristic functions: Englund (1980) uses
  - Berry–Esseen theorem for iid random variables,
  - Some technical adjustment to estimate the gaps amongst various normal distributions with different parameters.
- Coupling: challenging!
  - yet to see one bound using pure coupling but with the right order.
- Stein's method
  - No such work, have asked various people, from old to young!

Likely approximate distributions

- $N(t)$ is non-negative integer-valued, so if we consider d_{TV}, the approximate distribution must be non-negative integer-valued.
- If $\tau \sim \exp$, then $N(t)$ is Poisson (fixed point).
- If we use discretised normal, under what conditions can we get $N(t) \sim$ discretised normal for moderate t?
- Poisson(λ) is close to normal when λ is large.

Poisson is not enough!

- mean=variance, lack of flexibility.
- In general, for $a > 0$ and an integer b, a translated Poisson distribution is defined as $P_{a,b} := \mathrm{Pn}(a) * \delta_b$ (Röllin (2007)).
  - If N is close to a normal distribution, then it must be close to a translated Poisson distribution.
  - Suitable in the total variation distance so for all possible sets rather than intervals of the form $(-\infty, x]$.

Discretized normal

- $N^d(a+b, a)$ (Fang (2014)): having probability mass function at integer $z \in \mathbb{Z}$ as

$$\int_{z-1/2}^{z+1/2} \frac{1}{\sqrt{2\pi}a} e^{-\frac{(x-(a+b))^2}{2a^2}} dx.$$

  – Discretised normal can do the same job.

  – It does not offer the same interpretation as a translated Poisson.

Discrete CLT around $N(t)$

- Under the Kolmogorov distance, yes!
- Under d_{TV}: NO discrete CLT for $N(t)$!
  - If $\mathbb{P}(\tau = 1) = \mathbb{P}(\tau = 3) = 1/2$, then

$$\liminf_{t \to \infty} \min_{a,b} d_{\mathrm{TV}}(\mathscr{L}(N(t)), P_{a,b}) > 0.$$

  - Under what conditions can we have discrete CLT for $N(t)$?

The speed: $d_{\mathbf{W}}$

Let $a = t\sigma^2/\mu^3$ and $b = \lfloor \frac{t}{\mu}(1 - \frac{\sigma^2}{\mu^2})\rfloor$, where $\lfloor y \rfloor$ denotes the integer part of y. If $\mathbb{E}(\tau^3) < \infty$, then

$$d_{\mathrm{W}}(N(t), P_{a,b}) = O(1).$$

- If we standardise, then d_{W} is of order $O(t^{-1/2})$, hence Peccati, Solé, Taqqu & Utzet (2010) ensures

$$d_{\mathrm{W}}\left(\frac{N(t) - \mathbb{E}N(t)}{\sqrt{\mathrm{Var}(N(t))}}, N(0,1)\right) = O(t^{-1/2}).$$

The speed: $d_{\mathbf{TV}}$

- The Lebesgue decomposition theorem: for any distribution function G on $\mathbb{R}$ can be represented as

$$G = (1 - \alpha_G)G_s + \alpha_G G_a.$$

- A distribution function G on $\mathbb{R}$ is said to be non-singular if $\alpha_G > 0$.

The speed: $d_{\mathbf{TV}}$ (cont)

- a and b are as above. Assume $\mathbb{E}(\tau^3) < \infty$. Either of the following conditions ensures

$$d_{\mathrm{TV}}(N(t), P_{a,b}) = O\left(t^{-1/2}\right).$$

  - $0 < F(0) < 1$.
  - F is non-singular.
  - F is singular with $\mathrm{supp}(F) \cap \{c, 2c, 3c, \dots\} \neq \emptyset$ and $d_{\mathrm{TV}}(\tau, \tau + c) < 1$.

The Stein-Chen method

- $X \sim \mathrm{Pn}(\lambda)$ iff $\mathbb{P}(X+1=k) = \frac{k\mathbb{P}(X=k)}{\lambda}$, $k \in \mathbb{Z}_+$ iff $\mathbb{E}[\lambda g(X+1) - Xg(X)] = 0$ for a suitable class of g.

- Stein's identity for $\mathrm{Pn}(\lambda)$:

$$\lambda g(i+1) - ig(i) = f(i) - \mathrm{Pn}(\lambda)(f)$$

  for all suitable functions f.

  - $\mathrm{Pn}(\lambda)(f) = \mathbb{E}f(X)$ with $X \sim \mathrm{Pn}(\lambda)$.
  - By solving the equation recursively, g can be written in terms of f.

Stein's identity for $P_{a,b}$

Set $g(i) = 0$ for $i \le -1$ and write Stein's identity

$$\lambda g(i-b+1) - (i-b)g(i-b) = f(i-b) - \mathrm{Pn}(a)(f)$$

for all suitable functions f.

- There is a truncation problem at around $i = -1$.
- Write $\tilde{g}(j) := g(j-b)$ so that

$$\lambda \tilde{g}(i+1) - (i-b)\tilde{g}(i) \approx f(i-b) - \mathrm{Pn}(a)(f).$$

- Consider the stationary case $\mathcal{N}$ first: the first renewal needs adjustment.

Difficulty? a quick run of the proof

- By the Stein's identity for $P_{a,b}$:

$$
\begin{aligned}
& \mathbb{E}f(\mathcal{N}(t)-b)-\mathrm{Pn}(a)(f) \\
\approx \quad & a\mathbb{E}\tilde{g}(\mathcal{N}(t)+1)+b\mathbb{E}\tilde{g}(\mathcal{N}(t))-\mathbb{E}[\tilde{g}(\mathcal{N}(t))\mathcal{N}(t)]
\end{aligned}
$$

  – Need to work on $\mathbb{E}[\tilde{g}(\mathcal{N}(t))\mathcal{N}(t)]$.

From size biasing to Palm

- For a nonnegative integer-valued random variable X having positive finite mean μ, we consider $h(\cdot) = \delta_{\{k\}}(\cdot)$, then

$$\frac{\mathbb{E}[h(X)X]}{\mathbb{E}X} = \frac{k\mathbb{P}(X=k)}{\mu},$$

giving *size biased distribution*

$$\mathbb{P}(X^s = k) = \frac{k\mathbb{P}(X=k)}{\mu}.$$

From size biasing to Palm – cont

- $X^s \overset{\mathrm{d}}{=} X + 1$ iff $X \sim \mathrm{Pn}(\mu)$: the Stein-Chen method for Poisson approximation.
- Size biasing appears in various sampling contexts, e.g., in random digit dialing, it is proportionally more likely to dial households with more telephones than households with fewer phones.
- Size biasing of $\mathcal{N}(t)$ does not offer enough information.

From size biasing to Palm – cont

- We can expand

$$\mathbb{E}[\tilde{g}(\mathcal{N}(t))\mathcal{N}(t)] = \mathbb{E}\int_0^t \tilde{g}(\mathcal{N}(t))\mathcal{N}(d\alpha).$$

- It is possible to consider Radon-Nikodym derivative

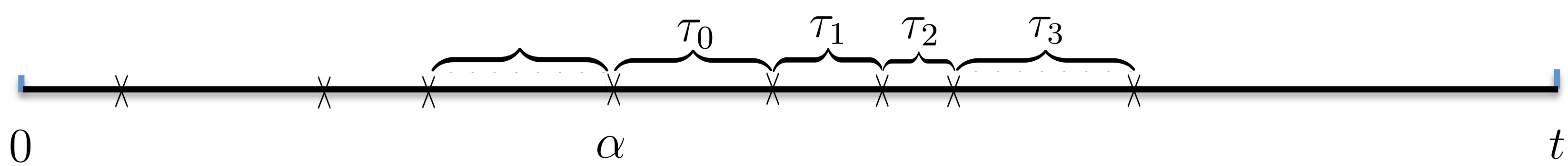
$$\frac{\mathbb{E}[\tilde{g}(\mathcal{N}(t))\mathcal{N}(d\alpha)]}{\mathbb{E}[\mathcal{N}(d\alpha)]} =: \mathbb{E}\tilde{g}(\mathcal{N}_\alpha(t)).$$

  – $\mathcal{N}_\alpha(t)$ is called a *Palm process* of $\mathcal{N}$ at α, its distribution is called the *Palm distribution*.

  – **Fact**: Palm distribution is the process version of size biasing.

The Palm for renewal process

- The Palm process at α: given there is a renewal at α, how the remaining part of the renewal process looks like?



- If $\mathcal{N}$ is a Poisson process, then $\mathcal{N}_\alpha \stackrel{\mathrm{d}}{=} \mathcal{N} + \delta_\alpha$: one additional observer at α, the rest is the same.

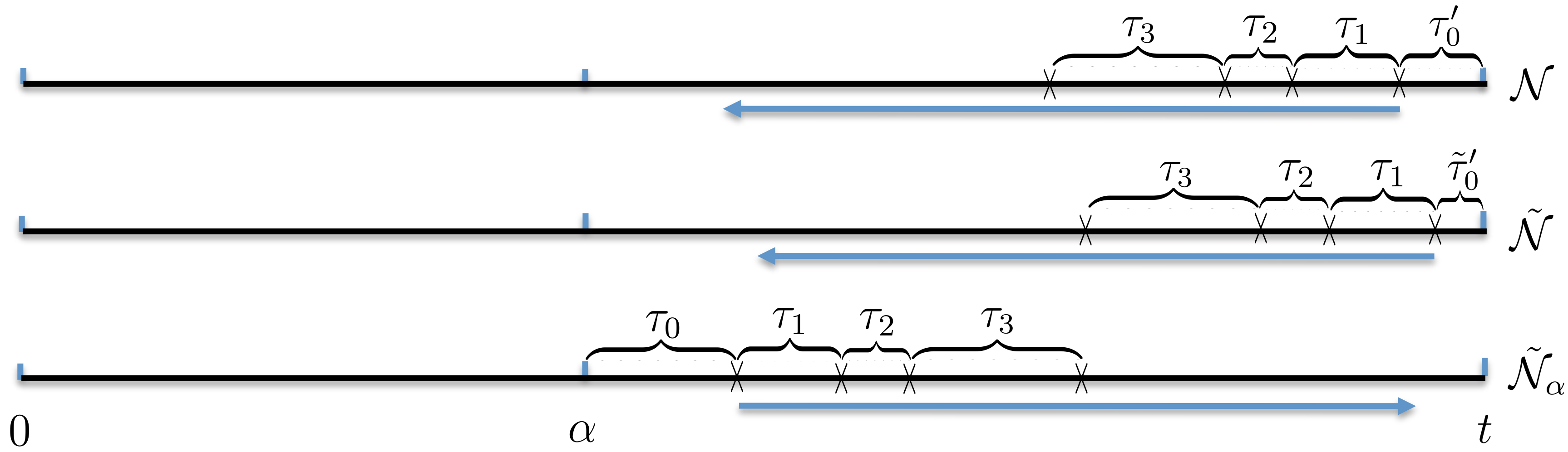
A quick run of the proof: cont

- Since $\mathbb{E}\mathcal{N}(d\alpha) = \frac{d\alpha}{\mu}$, we have

$$
\begin{aligned}
& \mathbb{E}f(\mathcal{N}(t)-b) - \mathrm{Pn}(a)(f) \\
\approx\ & a\mathbb{E}\tilde{g}(\mathcal{N}(t)+1) + b\mathbb{E}\tilde{g}(\mathcal{N}(t)) - \mathbb{E}\int_0^t \tilde{g}(\mathcal{N}(t))\mathcal{N}(d\alpha) \\
=\ & a\mathbb{E}\tilde{g}(\mathcal{N}(t)+1) + b\mathbb{E}\tilde{g}(\mathcal{N}(t)) - \frac{1}{\mu}\int_0^t \mathbb{E}\tilde{g}(\mathcal{N}_\alpha(t))d\alpha \\
=\ & a\mathbb{E}\Delta\tilde{g}(\mathcal{N}(t)) + (a+b)\mathbb{E}\tilde{g}(\mathcal{N}(t)) - \frac{1}{\mu}\int_0^t \mathbb{E}\tilde{g}(\mathcal{N}_\alpha(t))d\alpha \\
\approx\ & a\mathbb{E}\Delta\tilde{g}(\mathcal{N}(t)) - \frac{1}{\mu}\int_0^t \mathbb{E}[\tilde{g}(\mathcal{N}_\alpha(t)) - \tilde{g}(\mathcal{N}(t))]d\alpha,
\end{aligned}
$$

  - Δ is the forward difference operator and $a+b \approx t/\mu$,
  - $\mathcal{N}_\alpha$ is the Palm process of $\mathcal{N}$ at α.

A coupling that works!



Still unsatisfactory!

- Main difficulty in d_{TV}: $d_{\mathrm{TV}}(N(t), P_{a,b})$ has the same speed as $d_{\mathrm{TV}}(N(t), N(t)+1)$.
- What is $d_{\mathrm{TV}}(N(t), N(t)+1)$?
- If $\mathbb{P}(\tau = 1) = \mathbb{P}(\tau = 3) = 1/2$, then

$$\liminf_{t\to\infty} d_{\mathrm{TV}}(N(t), N(t)+1) \geq \sqrt{\frac{3}{8\pi}}e^{-8+O(t^{-1/2})}.$$

- Most regenerative events in Markov processes (both continuous and discrete time) have $d_{\mathrm{TV}}(N(t), P_{a,b}) = O(t^{-1/2})$.

Problems for further consideration

- If $\operatorname{supp}(F)$ contains an interval, then $d_{\mathrm{TV}}(N(t), P_{a,b}) = O(t^{-1/2})$.
- $d_{\mathrm{TV}}(N(t), P_{a,b}) = o(1)$ iff $d_{\mathrm{TV}}(N(t), P_{a,b}) = O(t^{-1/2})$.

Take home messages

For the distribution of the number of renewals approximated by a suitable translated Poisson (or discretized normal):

- d_{K}: order $O(t^{-1/2})$,
- d_{W}: $O(1)$,
- d_{TV}: $O(t^{-1/2})$ in most cases,
- the constants are too big and complicated.

Thank you!